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GRADIENT METHODS IN CONTROL THEORY
PART 5 - SEQUENTIAL GRADIENT-RESTORATION ALGORITHM: ADDITIONAL NUMERICAL EXAMPLES

by

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Gradient Methods in Control Theory

Part 5 - Sequential Gradient-Restoration Algorithm: Additional Numerical Examples[1]

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Abstract. In Ref. 1, Miele and Pritchard developed the sequential gradient-restoration algorithm for minimizing a functional subject to certain differential constraints and boundary conditions. In order to reach a more complete understanding of the properties of the sequential gradient-restoration algorithm, several modifications and extensions are studied. These modifications and extensions are concerned with (i) the scheme for updating the state, the control, and the parameter, (ii) the possibility of employing an incomplete restoration phase at the end of each gradient phase, and (iii) the search technique for the gradient stepsize. Several numerical examples are given.

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1. Introduction

In a previous report (Ref. 1), Miele and Pritchard developed the sequential-gradient restoration algorithm for minimizing a functional subject to certain differential constraints and boundary conditions. The algorithm includes the alternate succession of gradient phases and restoration phases. In the gradient phase, the first-order change of the functional is minimized subject to the linearized differential equations, the linearized boundary conditions, and a quadratic constraint on the variations of the control and the parameter. In the restoration phase, a functional quadratic in the variations of the control and the parameter is minimized subject to the linearized differential equations and the linearized boundary conditions.

In this report, several modifications and extensions of the sequential gradient-restoration algorithm are studied in order to reach a more complete understanding of the properties of the algorithm. These modifications and extensions are concerned with (i) the scheme for updating the state, the control, and the parameter, (ii) the possibility of employing an incomplete restoration phase at the end of each gradient phase, and (iii) the search technique for the gradient stepsize.

2. Statement of the Problem

The purpose of this paper is to study the minimization of the functional

$$I = \int_0^1 f(x, u, \pi, t)dt + [g(x, \pi)]_1 \tag{1}$$

with respect to the functions x(t), u(t), and the parameter π which satisfy the differential constraint

$$\dot{x} - \varphi(x, u, \pi, t) = 0 \tag{2}$$

the initial condition

$$(x)_0 = \text{given} \tag{3}$$

and the final condition

$$[\psi(x, \pi)]_1 = 0 \tag{4}$$

In the above equations, the functions f and g are scalar, the function φ is an n-vector, and the function ψ is a q-vector. The symbol x, an n-vector, denotes the state variable; the symbol u, an m-vector, denotes the control variable; and the symbol π, a p-vector, denotes the parameter. The time t, a scalar, is the independent variable; without loss of generality, the prescribed initial time is $t = 0$ and the prescribed final time is $t = 1$.

At the initial point, all the components of the state vector are given, so that $(x)_0$ is known. At the final point, q scalar relations are specified, where $0 \le q \le n + p$. Problems where the final time is other than unity can be reduced to the form (1)-(4) by normalizing the time with respect to the final time and by regarding the final time, if it is free, as one of the components of the parameter π.

3. Definitions

The following scalar quantities are now defined:

$$I = \int_0^1 f\,dt + (g)_1$$

$$L = \int_0^1 \lambda^T(\dot{x} - \varphi)dt + (\mu^T\psi)_1$$

$$J = I + L \tag{5}$$

$$P = \int_0^1 N(\dot{x} - \varphi)dt + N(\psi)_1$$

$$Q = \int_0^1 N(\dot{\lambda} - f_x + \varphi_x\lambda)dt + \int_0^1 N(f_u - \varphi_u\lambda)dt$$

$$+ N\left[\int_0^1 (f_\pi - \varphi_\pi\lambda)dt + (g_\pi + \psi_\pi\mu)_1\right] + N(\lambda + g_x + \psi_x\mu)_1$$

Here, I is the original integral, J the augmented integral, P the cumulative error in the constraints, and Q the cumulative error in the optimum conditions. Also, $\lambda(t)$, an n-vector, is a variable Lagrange multiplier and μ, a q-vector, is a constant Lagrange multiplier. The symbol N denotes the norm of a vector. The superscript T denotes the transpose of a matrix.

4. Basic Functions of the Sequential Gradient-Restoration Algorithm

Here, we summarize the sequential gradient-restoration algorithm of Ref. 1. We denote by $x(t)$, $u(t)$, π the conditions at the beginning of the gradient phase, by $\tilde{x}(t)$, $\tilde{u}(t)$, $\tilde{\pi}$ the conditions at the end of the gradient phase or beginning of the restoration phase, and by $\hat{x}(t)$, $\hat{u}(t)$, $\hat{\pi}$ the conditions at the end of the restoration phase. The variations leading from one set of conditions to another are defined by

$$\tilde{x}(t) = x(t) + \Delta x(t) = x(t) + \alpha A(t)$$

$$\tilde{u}(t) = u(t) + \Delta u(t) = u(t) + \alpha B(t) \tag{6}$$

$$\tilde{\pi} = \pi + \Delta\pi = \pi + \alpha C$$

and

$$\hat{x}(t) = \tilde{x}(t) + \Delta\tilde{x}(t) = \tilde{x}(t) + \tilde{\alpha}\tilde{A}(t)$$

$$\hat{u}(t) = \tilde{u}(t) + \Delta\tilde{u}(t) = \tilde{u}(t) + \tilde{\alpha}\tilde{B}(t) \tag{7}$$

$$\hat{\pi} = \tilde{\pi} + \Delta\tilde{\pi} = \tilde{\pi} + \tilde{\alpha}\tilde{C}$$

where α denotes the gradient stepsize and $\tilde{\alpha}$ denotes the restoration stepsize.

4.1. Gradient Phase. The functions $A(t)$, $B(t)$, C must be computed by solving the linear differential system

$$\dot{A} = \varphi_x^T A + \varphi_u^T B + \varphi_\pi^T C$$

$$\dot{\lambda} = f_x - \varphi_x \lambda$$

$$B = -f_u + \varphi_u \lambda \tag{8}$$

$$C = - \int_0^1 (f_\pi - \varphi_\pi \lambda) dt - (g_\pi + \psi_\pi \mu)_1$$

subject to the boundary conditions

$$(A)_0 = 0 \tag{9}$$

and

$$(\psi_x^T A + \psi_\pi^T C)_1 = 0 \quad , \quad (\lambda + g_x + \psi_x \mu)_1 = 0 \tag{10}$$

The solution of (8)-(10) can be obtained using the method of particular solutions developed by Miele in Ref. 2. For details relative to the implementation of the method, see Refs. 1-3.

4.2. Restoration Phase. The functions $\tilde{A}(t)$, $\tilde{B}(t)$, $\tilde{C}$ must be computed by solving the linear differential system

$$\dot{\tilde{A}} = \tilde{\varphi}_x^T \tilde{A} + \tilde{\varphi}_u^T \tilde{B} + \tilde{\varphi}_\pi^T \tilde{C} - (\dot{\tilde{x}} - \tilde{\varphi})$$

$$\dot{\tilde{\lambda}} = - \tilde{\varphi}_x \tilde{\lambda}$$

$$\tilde{B} = \tilde{\varphi}_u \tilde{\lambda} \tag{11}$$

$$\tilde{C} = \int_0^1 \tilde{\varphi}_\pi \tilde{\lambda} dt - (\tilde{\psi}_\pi \tilde{\mu})_1$$

subject to the boundary conditions

$$(\tilde{A})_0 = 0 \tag{12}$$

and

$$(\tilde{\psi} + \tilde{\psi}_x^T \tilde{A} + \tilde{\psi}_\pi^T \tilde{C})_1 = 0 \quad , \quad (\tilde{\lambda} + \tilde{\psi}_x \tilde{\mu})_1 = 0 \tag{13}$$

Once more, the solution of (11)-(13) can be obtained using the <u>method of particular solutions</u> (see Refs. 1-3).

5. Technical Aspects of the Sequential Gradient-Restoration Algorithm

In order to reach a more complete understanding of the properties of the sequential gradient-restoration algorithm several technical aspects are considered. They deal with (i) the scheme for updating the state, the control, and the parameter, (ii) the possibility of employing an incomplete restoration phase at the end of each gradient phase, and (iii) the search technique for the gradient stepsize.

5.1. Updating Technique. There are two basic techniques for updating the state, the control, and the parameter. They are called Scheme (a) and Scheme (b), respectively.

Scheme (a). For the gradient phase, compute $\tilde{x}(t)$, $\tilde{u}(t)$, $\tilde{\pi}$ according to Eqs. (6). For the restoration phase, compute $\hat{x}(t)$, $\hat{u}(t)$, $\hat{\pi}$ according to Eqs. (7).

Scheme (b). For the gradient phase, compute the varied control $\tilde{u}(t)$ and parameter $\tilde{\pi}$ from (6-2) and (6-3) and determine the varied state $\tilde{x}(t)$ by forward integration of the state differential equation (2) subject to the initial condition (3), that is, by forward integration of

$$\dot{x} = \varphi(x, \tilde{u}, \tilde{\pi}, t) \quad , \quad x(0) = \text{given} \tag{14}$$

For the restoration phase, compute the varied control $\hat{u}(t)$ and parameter $\hat{\pi}$ from (7-2) and (7-3) and determine the varied state $\hat{x}(t)$ by forward integration of the state differential equation (2) subject to the initial condition (3), that is, by forward integration of

$$\dot{x} = \varphi(x, \hat{u}, \hat{\pi}, t) \quad , \quad x(0) = \text{given} \tag{15}$$

5.2. Restoration Technique. There are two basic techniques for restoring the differential constraints and the boundary conditions at the end of each gradient phase: complete restoration and incomplete restoration.

Complete Restoration. The restoration phase of Section 4.2 consists of several cycles. In each cycle, the cumulative error in the constraints (5-4) is reduced; that is, the restoration stepsize $\tilde{\alpha}$ is chosen so that

$$\hat{P}(\tilde{\alpha}) < \hat{P}(0) \tag{16}$$

In practice, this can be achieved through a bisection process on $\tilde{\alpha}$ starting from $\tilde{\alpha} = 1$. The restoration cycle must be repeated several times until the cumulative error in the constraints (5-4) satisfies the inequalitity (for example)

$$\hat{P}(\tilde{\alpha}) \leq 10^{-8} \tag{17}$$

Incomplete Restoration. The restoration phase of Section 4.2 consists of a single cycle. In this cycle, the cumulative error in the constraints (5-4) is reduced in accordance with (16).

5.3. Search Technique for the Gradient Stepsize. In order to determine the optimum gradient stepsize α, a one-dimensional search on the functional $\tilde{Z}(\alpha)$ must be performed. Here, $\tilde{Z} = \tilde{I}$ or $\tilde{Z} = \tilde{J}$, depending on the choice. The search involves one-step quasilinearization or one-step cubic interpolation on $\tilde{Z}(\alpha)$, followed by a bisection process on α until the inequality

$$\tilde{Z}(\alpha) < \tilde{Z}(0) \tag{18}$$

is satisfied in combination with the further inequality (for example)

$$\begin{array}{ll} \text{Complete restoration} & \tilde{P}(\alpha) \leq 10 \\ \text{Incomplete restoration} & P(\alpha) \leq P(0) + 10 \end{array} \tag{19}$$

5.4. Remark. In the algorithm with complete restoration, the gradient stepsize α is acceptable providing

$$\hat{I} < I \tag{20}$$

where $\hat{I}$ is the value of the functional (1) at the end of the restoration phase and I is the value of the functional (1) at the beginning of the gradient phase. If Ineq. (20) is satisfied, the next gradient phase can be started. If Ineq. (20) is violated, one must return to the previous gradient phase and reduce the gradient stepsize α until, after restoration, Ineq. (20) is satisfied. This can be accomplished through successive bisections of α.

5.5. Stopping Condition. The sequential gradient-restoration is terminated when the following inequalities are satisfied simultaneously (for example):

$$P \leq 10^{-8} \quad , \quad Q \leq 10^{-4} \tag{21}$$

5.6. Starting Condition. The present algorithm can be started regardless of whether the nominal curve $x(t)$, $u(t)$, π satisfies the differential equation (2) and the boundary conditions (3)-(4). If the cumulative constraint error P satisfies Ineq. (21-1), the first gradient phase precedes the first restoration phase. If the cumulative constraint error P violates Ineq. (21-1), the first gradient phase must be preceded by a preliminary restoration phase performed in accordance with Section 5.2.

5.7. Bypassing Condition. At the end of the gradient phase, the cumulative constraint error $\tilde{P}$ must be computed. If $\tilde{P}$ violates Ineq. (21-1), the restoration phase must be started. If $\tilde{P}$ satisfies Ineq. (21-1), the restoration phase is bypassed and the next gradient phase is started.

6. Sample Problems

The following sample problems, already considered in Refs. 4-5, are investigated here. Scalar notations are employed.

Problem 6.1. Consider the problem of minimizing the integral

$$I = \int_0^1 (1 + x^2 + y^2 + u^2) dt \tag{22}$$

subject to the differential constraints

$$\dot{x} = u - y^2 \quad , \quad \dot{y} = u - xy \tag{23}$$

and the boundary conditions

$$\begin{aligned} x(0) &= 0 \quad , \quad x(1) = 1 \\ y(0) &= 1 \quad , \quad y(1) = 2 \end{aligned} \tag{24}$$

Scheme (a). Assume the nominal functions ($N = -1$)

$$x(t) = t \quad , \quad y(t) = 1 + t \quad , \quad u(t) = 1 \tag{25}$$

which are consistent with the boundary conditions (24) but not consistent with the differential constraints (23).

Scheme (b). Assume the nominal control ($N = -1$)

$$u(t) = 1 \tag{26}$$

and the nominal state $x(t)$, $y(t)$ obtained by forward integration of Eqs. (23) subject to

(26) and the initial conditions

$$x(0) = 0 \quad , \quad y(0) = 1 \tag{27}$$

Problem 6.2. Consider the problem of maximizing the integral

$$I = 2\int_0^1 \cos u dt \tag{28}$$

subject to the differential constraints

$$\dot{x} = 2 \sin u - 1, \quad \dot{y} = x \tag{29}$$

and the boundary conditions

$$\begin{aligned} x(0) &= 0 \quad , \quad x(1) = 0 \\ y(0) &= 0 \quad , \quad y(1) = 0.3 \end{aligned} \tag{30}$$

Scheme (a). Assume the nominal functions (N = -1)

$$x(t) = 0 \quad , \quad y(t) = 0.3t \quad , \quad u(t) = 0 \tag{31}$$

which are consistent with the boundary conditions (30) but not consistent with the differential constraints (29).

Scheme (b). Assume the nominal control (N = -1)

$$u(t) = 0 \tag{32}$$

and the nominal state x(t), y(t) obtained by forward integration of Eqs. (29) subject to (32) and the initial conditions

$$x(0) = 0 \quad , \quad y(0) = 0 \tag{33}$$

Problem 6.3. Consider the problem of minimizing the functional

$$I = \tau \tag{34}$$

subject to the differential constraints

$$\dot{x} = \tau u \quad , \quad \dot{y} = \tau(x^2 - u^2) \tag{35}$$

and the boundary conditions

$$\begin{aligned} x(0) = 0 \quad &, \quad x(1) = 1 \\ y(0) = 0 \quad &, \quad y(1) = 0 \end{aligned} \tag{36}$$

Scheme (a). Assume the nominal functions (N = -1)

$$x(t) = t \quad , \quad y(t) = 0 \quad , \quad u(t) = 1 \quad , \quad \tau = 1 \tag{37}$$

which are consistent with the boundary conditions (36) but not consistent with the differential constraints (35).

Scheme (b). Assume the nominal control and parameter (N = -1)

$$u(t) = 1 \quad , \quad \tau = 1 \tag{38}$$

and the nominal state x(t), y(t) obtained by forward integration of Eqs. (35) subject to (38) and the initial conditions

$$x(0) = 0 \quad , \quad y(0) = 0 \tag{39}$$

Problem 6.4. Consider the problem of minimizing the functional

$$I = \tau \tag{40}$$

subject to the differential constraints

$$\dot{x} = \tau z \cos u \quad , \quad \dot{y} = \tau z \sin u \quad , \quad \dot{z} = \tau \sin u \tag{41}$$

and the boundary conditions

$$\begin{aligned} x(0) &= 0 \quad , \quad x(1) = 1 \\ y(0) &= 0 \quad , \quad x(1) = \text{free} \\ z(0) &= 0 \quad , \quad z(1) = \text{free} \end{aligned} \tag{42}$$

Scheme (a). Assume the nominal functions $(N = -1)$

$$x(t) = t \quad , \quad y(t) = 0 \quad , \quad z(t) = 0 \quad , \quad u(t) = 1 \quad , \quad \tau = 1 \tag{43}$$

which are consistent with the boundary conditions (42) but not consistent with the differential constraints (41).

Scheme (b). Assume the nominal control and parameter $(N = -1)$

$$u(t) = 1 \quad , \quad \tau = 1 \tag{44}$$

and the nominal state $x(t)$, $y(t)$, $z(t)$ obtained by forward integration of Eqs. (41) subject to (44) and the initial conditions

$$x(0) = 0 \quad , \quad y(0) = 0 \quad , \quad z(0) = 0 \tag{45}$$

7. Numerical Experiments

The computations reported here were performed on the Rice University B-5500 computer in double-precision arithmetic. The algorithm was programmed in FORTRAN IV. The interval of integration was divided into 50 steps. The differential systems (8)-(10) and (11)-(13) were integrated using Hamming's modified predictor-corrector method with a special Runge-Kutta procedure to start the integration routine (Ref. 6). The definite integrals $\bar{I}$, L, J, P, Q were computed using Simpson's rule.

The detailed results are presented in Tables 1-18 where the following terminology is employed: N is the iteration number, N_G is the number of gradient cycles per iteration, N_S is the number of times the gradient stepsize is bisected per iteration, N_R is the number of restoration cycles per iteration, I is the original integral (1), P is the cumulative error in the constraints (5-4), and Q is the cumulative error in the optimum conditions (5-5).

Summary results are presented in Tables 19-22, where the following terminology is employed: N_* is the total number of iterations for convergence, ΣN_G is the total number of gradient cycles, ΣN_S is the total number of bisections of the gradient stepsize, and ΣN_R is the total number of restoration cycles. To a first approximation, each gradient cycle and each restoration cycle require the same computing time. Therefore, the cumulative number of gradient cycles and restoration cycles $\Sigma N_G + \Sigma N_R$ is proportional to the overall computing time and, hence, is a measure of the efficiency of a computing scheme.

(i) Updating Technique. In Table 19, the updating Scheme (a) is compared with the updating Scheme (b). In these experiments, complete restoration was required.

The search to determine the optimum gradient stepsize was performed on the augmented functional $\tilde{J}(\alpha)$ using one-step cubic interpolation.

For Problems 6.1 through 6.4, Scheme (a) is superior to Scheme (b) in that it requires a smaller cumulative number of cycles $\Sigma N_G + \Sigma N_R$. In addition, the one-dimensional search for determining the gradient stepsize or the restoration stepsize is easier with Scheme (a), since this scheme makes full use of the information obtained when solving the linearized, two-point boundary-value problem associated with either the gradient phase or the restoration phase. For these combined reasons, the computer time for convergence is smaller for Scheme (a) than for Scheme (b).

(ii) Restoration Technique. In Table 20, the complete restoration algorithm is compared with the incomplete restoration algorithm. In these experiments, the updating Scheme (a) was used. The search to determine the optimum gradient stepsize was performed on the augmented functional $\tilde{J}(\alpha)$ using one-step quasilinearization.

For Problems 6.1 and 6.3, the complete restoration algorithm and the incomplete restoration algorithm require the same cumulative number of cycles $\Sigma N_G + \Sigma N_R$. For Problem 6.2, the algorithm with incomplete restoration is superior to the algorithm with complete restoration, while the opposite occurs in Problem 6.4.

While these examples are somewhat inconclusive, the author believes that the algorithm with complete restoration should be preferred in engineering applications. This is because it produces a succession of intermediate solutions $x(t)$, $u(t)$, π which are physically possible, that is, a succession of functions consistent with the differential constraint (2), the initial condition (3), and the final condition (4). At any rate, the algorithm with incomplete restoration looks interesting and, certainly, a further investigation is warranted.

(iii) Search Technique. In Table 21, the algorithm with search on $\tilde{J}(\alpha)$ is compared with the algorithm with search on $\tilde{I}(\alpha)$. In these experiments, the updating Scheme (a) was used. Complete restoration was required. The search to determine the optimum gradient stepsize was performed with one-step quasilinearization.

For Problems 6.1 and 6.2, the algorithm with search on $\tilde{J}(\alpha)$ is superior to the algorithm with search on $\tilde{I}(\alpha)$. This is because the cumulative number of cycles $\Sigma N_G + \Sigma N_R$ is smaller.

For Problems 6.3 and 6.4, the results are not given. Since the functional $\tilde{I}(\alpha)$ is linear in α, the search on $\tilde{I}(\alpha)$ fails to predict an optimum value for the stepsize. For these problems, the use of the functional $\tilde{J}(\alpha)$ is indispensable.

(iv) Search Technique. In Table 22, the algorithm with search via quasilinearization is compared with the algorithm with search via cubic interpolation. In these experiments, the updating Scheme (a) was used. Complete restoration was required. The search to determine the optimum gradient stepsize was performed on the augmented functional $\tilde{J}(\alpha)$.

For Problems 6.1, 6.2, 6.3, the algorithm with search by quasilinearization and the algorithm with search by cubic interpolation require the same cumulative number of cycles. For Problem 6.4, the algorithm with search by quasilinearization is slightly superior to the algorithm with search by cubic interpolation. Therefore, if second derivatives cannot be used, cubic interpolation can be employed as an efficient alternative to quasilinearization.

Table 1. Problem 6.1.

Updating Scheme (a), complete restoration, search via quasilinearization on $\tilde{J}(\alpha)$.

N	N_G	N_S	N_R	I	P	Q
-1				0.46666×10^{1}	0.7×10^{1}	
0			4	0.33677×10^{2}	0.3×10^{-10}	0.9×10^{0}
1	1	0	1	0.33466×10^{2}	0.8×10^{-13}	0.5×10^{-2}
2	1	0	1	0.33464×10^{2}	0.5×10^{-9}	0.4×10^{-4}

Table 2. Problem 6.1.

Updating Scheme (a), complete restoration, search via quasilinearization on $\tilde{I}(\alpha)$.

N	N_G	N_S	N_R	I	P	Q
-1				0.46666×10^{1}	0.7×10^{1}	
0			4	0.33677×10^{2}	0.3×10^{-10}	0.9×10^{0}
1	1	0	1	0.33468×10^{2}	0.1×10^{-12}	0.1×10^{-1}
2	1	0	1	0.33465×10^{2}	0.5×10^{-20}	0.4×10^{-3}
3	1	0	0	0.33465×10^{2}	0.1×10^{-10}	0.1×10^{-4}

Table 3. Problem 6.1.

Updating Scheme (a), complete restoration, search via cubic interpolation on $\tilde{J}(\alpha)$.

N	N_G	N_S	N_R	I	P	Q
-1				0.46666×10^{1}	0.7×10^{1}	
0			4	0.33677×10^{2}	0.3×10^{-10}	0.9×10^{0}
1	1	0	1	0.33466×10^{2}	0.8×10^{-13}	0.5×10^{-2}
2	1	0	1	0.33464×10^{2}	0.4×10^{-9}	0.4×10^{-4}

Table 4. Problem 6.1.

Updating Scheme (b), complete restoration, search via cubic interpolation on $\tilde{J}(\alpha)$.

N	N_G	N_S	N_R	I	P	Q
-1				0.59813×10^{1}	0.1×10^{2}	
0			5	0.34274×10^{2}	0.1×10^{-15}	0.3×10^{1}
1	1	0	2	0.33472×10^{2}	0.3×10^{-15}	0.3×10^{-1}
2	1	0	1	0.33465×10^{2}	0.2×10^{-16}	0.2×10^{-3}
3	1	0	0	0.33465×10^{2}	0.2×10^{-11}	0.1×10^{-5}

Table 5. Problem 6.1.

Updating Scheme (a), incomplete restoration, search via quasilinearization on $\tilde{J}(\alpha)$.

N	N_G	N_S	N_R	I	P	Q
-1				0.46666×10^{1}	0.7×10^{1}	
0			1	0.89609×10^{1}	0.4×10^{1}	0.1×10^{0}
1	1	0	1	0.29994×10^{2}	0.1×10^{1}	0.9×10^{0}
2	1	0	1	0.33397×10^{2}	0.6×10^{-3}	0.5×10^{-1}
3	1	0	1	0.33465×10^{2}	0.3×10^{-10}	0.1×10^{-3}
4	1	0	0	0.33465×10^{2}	0.4×10^{-10}	0.1×10^{-5}

Table 6. Problem 6.2.

Updating Scheme (a), complete restoration, search via quasilinearization on $\tilde{J}(\alpha)$.

N	N_G	N_S	N_R	I	P	Q
-1				0.20000×10^{1}	0.1×10^{1}	
0			4	0.11166×10^{1}	0.1×10^{-8}	0.6×10^{0}
1	1	0	2	0.11653×10^{1}	0.2×10^{-11}	0.3×10^{-1}
2	1	0	1	0.11692×10^{1}	0.1×10^{-10}	0.3×10^{-2}
3	1	0	1	0.11695×10^{1}	0.6×10^{-15}	0.6×10^{-3}
4	1	0	0	0.11696×10^{1}	0.2×10^{-8}	0.1×10^{-3}
5	1	0	0	0.11696×10^{1}	0.3×10^{-8}	0.4×10^{-4}

Table 7. Problem 6.2.

Updating Scheme (a), complete restoration, search via quasilinearization on $\tilde{I}(\alpha)$.

N	N_G	N_S	N_R	I	P	Q
-1				0.20000×10^{1}	0.1×10^{1}	
0			4	0.11166×10^{1}	0.1×10^{-8}	0.6×10^{0}
1	1	1	6	0.11355×10^{1}	0.3×10^{-11}	0.3×10^{0}
2	1	2	9	0.11637×10^{1}	0.3×10^{-11}	0.8×10^{-1}
3	1	3	8	0.11692×10^{1}	0.9×10^{-9}	0.5×10^{-2}
4	1	3	5	0.11694×10^{1}	0.8×10^{-13}	0.1×10^{-2}
5	1	4	6	0.11695×10^{1}	0.3×10^{-16}	0.2×10^{-4}

Table 8. Problem 6.2.

Updating Scheme (a), complete restoration, search via cubic interpolation on $\tilde{J}(\alpha)$.

N	N_G	N_S	N_R	I	P	Q
-1				0.20000×10^{1}	0.1×10^{1}	
0			4	0.11166×10^{1}	0.1×10^{-8}	0.6×10^{0}
1	1	0	2	0.11651×10^{1}	0.1×10^{-11}	0.3×10^{-1}
2	1	0	1	0.11692×10^{1}	0.1×10^{-10}	0.3×10^{-2}
3	1	0	1	0.11695×10^{1}	0.5×10^{-15}	0.6×10^{-3}
4	1	0	0	0.11696×10^{1}	0.2×10^{-8}	0.1×10^{-3}
5	1	0	0	0.11696×10^{1}	0.3×10^{-8}	0.5×10^{-4}

Table 9. Problem 6.2.

Updating Scheme (b), complete restoration, search via cubic interpolation on $\tilde{J}(\alpha)$.

N	N_G	N_S	N_R	I	P	Q
-1				0.20000×10^{1}	0.1×10^{1}	
0			4	0.11166×10^{1}	0.7×10^{-9}	0.6×10^{0}
1	1	0	2	0.11651×10^{1}	0.1×10^{-11}	0.3×10^{-1}
2	1	0	1	0.11692×10^{1}	0.1×10^{-10}	0.3×10^{-2}
3	1	0	0	0.11697×10^{1}	0.8×10^{-8}	0.6×10^{-3}
4	1	1	1	0.11697×10^{1}	0.9×10^{-8}	0.2×10^{-3}
5	1	2	2	0.11698×10^{1}	0.9×10^{-8}	0.1×10^{-3}
6	1	3	3	0.11698×10^{1}	0.9×10^{-8}	0.9×10^{-4}

Table 10. Problem 6.2.

Updating Scheme (a), incomplete restoration, search via quasilinearization on $\tilde{J}(\alpha)$.

-1				0.20000×10^{1}	0.1×10^{1}	
0			1	0.15276×10^{1}	0.7×10^{-1}	0.1×10^{0}
1	1	0	1	0.12603×10^{1}	0.3×10^{-2}	0.1×10^{0}
2	1	0	1	0.11773×10^{1}	0.3×10^{-4}	0.3×10^{-1}
3	1	0	1	0.11693×10^{1}	0.1×10^{-7}	0.2×10^{-2}
4	1	0	1	0.11695×10^{1}	0.1×10^{-13}	0.5×10^{-3}
5	1	0	0	0.11696×10^{1}	0.4×10^{-9}	0.6×10^{-4}

Table 11. Problem 6.3.

Updating Scheme (a), complete restoration, search via quasilinearization on $\tilde{J}(\alpha)$.

N	N_G	N_S	N_R	$I = \tau$	P	Q
-1				0.10000×10^{1}	0.5×10^{0}	
0		-	4	0.15810×10^{1}	0.7×10^{-16}	0.5×10^{-1}
1	1	0	1	0.15708×10^{1}	0.2×10^{-8}	0.2×10^{-3}
2	1	0	0	0.15707×10^{1}	0.3×10^{-8}	0.1×10^{-5}

Table 12. Problem 6.3.

Updating Scheme (a), complete restoration, search via cubic interpolation on $\tilde{J}(\alpha)$.

N	N_G	N_S	N_R	$I = \tau$	P	Q
-1				0.10000×10^{1}	0.5×10^{0}	
0			4	0.15810×10^{1}	0.7×10^{-16}	0.5×10^{-1}
1	1	0	1	0.15708×10^{1}	0.3×10^{-8}	0.1×10^{-3}
2	1	0	0	0.15707×10^{1}	0.2×10^{-8}	0.1×10^{-5}

Table 13. Problem 6.3.

Updating Scheme (b), complete restoration, search via cubic interpolation on $\tilde{J}(\alpha)$.

N	N_G	N_S	N_R	$I = \tau$	P	Q
-1				0.10000×10^{1}	0.4×10^{0}	
0			4	0.15877×10^{2}	0.1×10^{-11}	0.8×10^{-1}
1	1	0	2	0.15708×10^{2}	0.1×10^{-16}	0.3×10^{-3}
2	1	0	0	0.15707×10^{2}	0.3×10^{-8}	0.4×10^{-5}

Table 14. Problem 6.3.

Updating Scheme (a), incomplete restoration, search via quasilinearization on $\tilde{J}(\alpha)$.

N	N_G	N_S	N_R	$I = \tau$	P	Q
-1				0.10000×10^{1}	0.5×10^{0}	
0			1	0.14545×10^{1}	0.7×10^{-1}	0.2×10^{-1}
1	1	0	1	0.15696×10^{1}	0.6×10^{-3}	0.1×10^{-1}
2	1	0	1	0.15706×10^{1}	0.1×10^{-7}	0.9×10^{-4}
3	1	0	1	0.15707×10^{1}	0.1×10^{-15}	0.5×10^{-6}

Table 15. Problem 6.4.

Updating Scheme (a), complete restoration, search via quasilinearization on $\tilde{J}(\alpha)$.

N	N_G	N_S	N_R	$I = \tau$	P	Q
-1				0.10000×10^1	0.1×10^1	
0			5	0.18337×10^1	0.4×10^{-16}	0.2×10^0
1	1	0	2	0.17728×10^1	0.1×10^{-11}	0.1×10^{-2}
2	1	0	1	0.17724×10^1	0.8×10^{-15}	0.1×10^{-4}

Table 16. Problem 6.4.

Updating Scheme (a), complete restoration search via cubic interpolation on $\tilde{J}(\alpha)$.

N	N_G	N_S	N_R	$I = \tau$	P	Q
-1				0.10000×10^1	0.1×10^1	
0			5	0.18337×10^1	0.4×10^{-16}	0.2×10^0
1	1	0	2	0.17826×10^1	0.2×10^{-9}	0.4×10^{-1}
2	1	0	1	0.17726×10^1	0.3×10^{-9}	0.6×10^{-3}
3	1	0	1	0.17724×10^1	0.1×10^{-16}	0.9×10^{-5}

Table 17. Problem 6.4.

Updating Scheme (b), complete restoration,search via cubic interpolation on $\tilde{J}(\alpha)$.

N	N_G	N_S	N_R	$I = \tau$	P	Q
-1				0.10000×10^{1}	0.5×10^{0}	
0			5	0.19402×10^{1}	0.4×10^{-10}	0.3×10^{0}
1	1	1	7	0.17766×10^{1}	0.4×10^{-14}	0.1×10^{-1}
2	1	0	1	0.17725×10^{1}	0.3×10^{-10}	0.1×10^{-3}
3	1	0	0	0.17724×10^{1}	0.2×10^{-8}	0.3×10^{-5}

Table 18. Problem 6.4.

Updating Scheme (a), incomplete restoration, search via quasilinearization on $\tilde{J}(\alpha)$.

N	N_G	N_S	N_R	$I = \tau$	P	Q
-1				0.10000×10^{1}	0.1×10^{1}	
0			1	0.15483×10^{1}	0.1×10^{1}	0.7×10^{0}
1	1	0	1	0.36702×10^{0}	0.3×10^{1}	0.8×10^{0}
2	1	0	1	0.12611×10^{1}	0.2×10^{1}	0.9×10^{0}
3	1	0	1	0.10863×10^{1}	0.2×10^{1}	0.6×10^{0}
4	1	0	1	0.16967×10^{1}	0.7×10^{0}	0.2×10^{0}
5	1	0	1	0.17202×10^{1}	0.8×10^{-1}	0.1×10^{0}
6	1	0	1	0.17979×10^{1}	0.2×10^{-3}	0.3×10^{-1}
7	1	0	1	0.17838×10^{1}	0.3×10^{-5}	0.3×10^{-1}
8	1	0	1	0.17714×10^{1}	0.4×10^{-9}	0.1×10^{-2}
9	1	0	1	0.17710×10^{1}	0.2×10^{-15}	0.2×10^{-4}

Table 19. Comparison of updating techniques.

Problem	Symbol	Scheme (a)	Scheme (b)
6.1	N_*	2	3
	ΣN_G	2	3
	ΣN_S	0	0
	ΣN_R	6	8
	$\Sigma N_G + \Sigma N_R$	8	11
6.2	N_*	5	6
	ΣN_G	5	6
	ΣN_S	0	6
	ΣN_R	8	13
	$\Sigma N_G + \Sigma N_R$	13	19
6.3	N_*	2	2
	ΣN_G	2	2
	ΣN_S	0	0
	ΣN_R	5	6
	$\Sigma N_G + \Sigma N_R$	7	8
6.4	N_*	3	3
	ΣN_G	3	3
	ΣN_S	0	1
	ΣN_R	9	13
	$\Sigma N_G + \Sigma N_R$	12	16

Table 20. Comparison of restoration techniques.

Problem	Symbol	Complete restoration	Incomplete restoration
6.1	N_*	2	4
	ΣN_G	2	4
	ΣN_S	0	0
	ΣN_R	6	4
	$\Sigma N_G + \Sigma N_R$	8	8
6.2	N_*	5	5
	ΣN_G	5	5
	ΣN_S	0	0
	ΣN_R	8	5
	$\Sigma N_G + \Sigma N_R$	13	10
6.3	N_*	2	3
	ΣN_G	2	3
	ΣN_S	0	0
	ΣN_R	5	4
	$\Sigma N_G + \Sigma N_R$	7	7
6.4	N_*	2	9
	ΣN_G	2	9
	ΣN_S	0	0
	ΣN_R	8	10
	$\Sigma N_G + \Sigma N_R$	10	19

Table 21. Comparison of search techniques.

Problem	Symbol	Search on $\tilde{J}(\alpha)$	Search on $\tilde{I}(\alpha)$
6.1	N_*	2	3
	ΣN_G	2	3
	ΣN_S	0	0
	ΣN_R	6	6
	$\Sigma N_G + \Sigma N_R$	8	9
6.2	N_*	5	5
	ΣN_G	5	5
	ΣN_S	0	13
	ΣN_R	8	38
	$\Sigma N_G + \Sigma N_R$	13	43

Table 22. Comparison of search techniques.

Problem	Symbol	Quasilinearization	Cubic interpolation
6.1	N_*	2	2
	ΣN_G	2	2
	ΣN_S	0	0
	ΣN_R	6	6
	$\Sigma N_G + \Sigma N_R$	8	8
6.2	N_*	5	5
	ΣN_G	5	5
	ΣN_S	0	0
	ΣN_R	8	8
	$\Sigma N_G + \Sigma N_R$	13	13
6.3	N_*	2	2
	ΣN_G	2	2
	ΣN_S	0	0
	ΣN_R	5	5
	$\Sigma N_G + \Sigma N_R$	7	7
6.4	N_*	2	3
	ΣN_G	2	3
	ΣN_S	0	0
	ΣN_R	8	9
	$\Sigma N_G + \Sigma N_R$	10	12

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